Math 115A - Spring 2019
Practice Exam 1 - Solutions

Full Name: ______________________________

UID: ______________________________

Instructions:

- Read each problem carefully.
- Show all work clearly and circle or box your final answer where appropriate.
- Justify your answers. A correct final answer without valid reasoning will not receive credit.
- All work including proofs should be well organized and clearly written using complete sentences.
- You may use the provided scratch paper, however this work will not be graded unless very clearly indicated there and in the exam.
- Calculators are not allowed but you may have a 3×5 inch notecard.

Page	Points	Score
1	10	
2	15	
3	10	
4	10	
Total:	45	

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1. (10 points) True or False: Prove or disprove the following statements.

   (a) If U_1, U_2, and W are subspaces of a finite-dimensional vector space V such that $U_1 + W = U_2 + W$, then $U_1 = U_2$.

   (b) Fix an $n \times n$ matrix B and let $W = \{A \in M_{n\times n}(\mathbb{F}) \mid AB = BA\}$. Then W is a subspace of $M_{n\times n}(\mathbb{F})$.

Solution:

(a) **False.**

Take $V = \mathbb{R}^2$ and let $U_1 = \text{span}\{(1,0)\}$, $U_2 = \text{span}\{(0,1)\}$ and $W = \text{span}\{(1,1)\}$. Then $U_1 + W = U_2 + W = \mathbb{R}^2$ but $U_1 \neq U_2$.

(b) **True.**

Proof. To show that W is a subspace we need to check that W is closed under addition and scalar multiplication, and that W contains the zero vector. Fix B and let M and N be matrices in W so that $MB = BM$ and $NB = BN$. Then

$$(M+N)B = MB + NB = BM + BN = B(M+N)$$

so $M + N \in W$. Let $\lambda \in \mathbb{F}$. Then $\lambda M \in W$ since

$$(\lambda M)B = \lambda(MB) = \lambda(BM) = B(\lambda M).$$

Finally, in $M_{n\times n}(\mathbb{F})$ the zero vector is the zero matrix and $0B = 0 = B0$ so $0 \in W$. Thus W is a subspace of $M_{n\times n}(\mathbb{F})$. □

2. (15 points) True or False: Prove or disprove the following statements.

(a) The set $W = \{(a,b,c) \in \mathbb{R}^3 \mid a^2+b^2+c^2=0\}$ is a subspace of $\mathbb{R}^3$.

(b) The set $W = \{(a,b,c) \in \mathbb{R}^3 \mid a+b+c=0\}$ is a subspace of $\mathbb{R}^3$.

(c) There exists a linear transformation $T : \mathbb{F}^5 \to \mathbb{F}^2$ with

$$\ker T = \{(a,b,c,d,e) \in \mathbb{F}^5 \mid a=b \text{ and } c=d=e\}.$$

Solution:

(a) **True.**

Proof. Let $a,b,c \in \mathbb{R}$ with $a^2+b^2+c^2=0$. Since $a^2, b^2, c^2 \geq 0$, it must be that $a=b=c=0$. So $W=\{0\}$, which is subspace. □

(b) **True.**

Proof. In order to show W is a subspace we check that W is closed under addition and scalar multiplication, and contains the zero vector. Given two arbtirary elements of W, say (a,b,c) and $(\bar{a},\bar{b},\bar{c})$, so that $a+b+c=0$ and $\bar{a}+\bar{b}+\bar{c}=0$, we want to show their sum is in W. We compute

$$(a,b,c)+(\bar{a},\bar{b},\bar{c}) = (a+\bar{a}, b+\bar{b}, c+\bar{c}).$$

The sum is in W since

$$(a+\bar{a})+(b+\bar{b})+(c+\bar{c}) = (a+b+c)+(\bar{a}+\bar{b}+\bar{c}) = 0+0=0.$$

So W is closed under addition. Now for scalar multiplication, given $\lambda \in \mathbb{R}$ we need that $\lambda(a,b,c) \in W$. This follows because

$$\lambda(a,b,c) = (\lambda a, \lambda b, \lambda c)$$

and

$$\lambda a + \lambda b + \lambda c = \lambda(a+b+c) = \lambda 0 = 0.$$

Last, we check that $(0,0,0) \in W$, but of course $0+0+0=0$. Thus W is a subspace of $\mathbb{R}^3$. □

(c) **False.**

By the Rank-Nullity Theorem, $\dim(\ker T) + \dim(\operatorname{im} T) = \dim \mathbb{F}^5 = 5$. But we see that $\dim(\ker T)$ has dimension 2 since $\{(1,1,0,0,0),(0,0,1,1,1)\}$ gives a basis for $\ker T$. This implies that $\dim(\operatorname{im} T) = 3$. But $\operatorname{im} T$ is a subspace of $\mathbb{F}^2$ so $\dim(\operatorname{im} T) \leq 2$, a contradiction.

3. (10 points) True or False: Prove or disprove the following statements.

(a) Let $S = \{(1,-1,0),(0,1,-1),(1,1,1)\} \subseteq \mathbb{R}^3$. The list S is a basis for $\mathbb{R}^3$.

(b) Let $B = \{(1,-1,0),(0,1,-1),(1,1,1)\} \subseteq (\mathbb{F}_2)^3$. The list B is a basis for $(\mathbb{F}_2)^3$.

Solution:

(a) **True.**

Proof. Since the dimension of $\mathbb{R}^3$ is 3 and S has 3 elements, it suffices to prove either that S is linearly independent or that $\operatorname{span} S = \mathbb{R}^3$, because one will imply the other. We will prove that S is linearly independent. Consider a linear combination

$$a(1,-1,0) + b(0,1,-1) + c(1,1,1) = (a+c, -a+b+c, -b+c) = 0$$

with scalars $a, b, c \in \mathbb{R}$. This gives a system of linear equations

$$\begin{aligned} a + c &= 0 \\ -a + b + c &= 0 \\ -b + c &= 0. \end{aligned}$$

We will show that $a = b = c = 0$. Adding b to both sides of the last equation gives $b = c$. So the first two equations become

$$\begin{aligned} a + b &= 0 \\ -a + 2b &= 0 \end{aligned}$$

Adding a to both sides of the second equation now gives $a = 2b$. But then the first equation becomes $3b = 0$. Hence $b = 0$ and then also $c = 0$ and $a = 0$. Thus there are no nontrivial linear combinations of zero and S is linearly independent. Since $\mathbb{R}^3$ has dimension 3, this shows S is a basis for $\mathbb{R}^3$. □

(b) **True.**

Proof. We have seen that sometimes a basis for $\mathbb{R}^3$ is not a basis for $(\mathbb{F}_2)^3$. However, in this case the same argument as above holds (though we can now ignore the minus signs), because $3 = 1 \in \mathbb{F}_2$. In fact the argument could be shorter, because once we have $a = 2b$, we know $a = 0$ since $2 = 0 \in \mathbb{F}_2$. But then $a = b = c = 0$ and B is linearly independent. Since $\mathbb{F}^3$ has dimension 3 and B contains 3 linearly independent vectors, B also spans. Hence B is a basis for $(\mathbb{F}_2)^3$. □

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4. (10 points) True or False: Let W_1 and W_2 be subspaces of a vector space V over a field $\mathbb{F}$. Prove or disprove the following sets are subspaces of V.

(a) The intersection of W_1 and W_2, given by

$$W_1 \cap W_2 = \{v \in V \mid v \in W_1 \text{ and } v \in W_2\}.$$

(b) The difference of W_1 from W_2, given by

$$W_2 - W_1 = \{v \in V \mid v \in W_2 \text{ and } v \notin W_1\}.$$

Solution:

(a) **True.**

Proof. We need to show that $W_1 \cap W_2$ is closed under addition and scalar multiplication, and that it contains $0 \in V$. All of these follow from the fact that W_1 and W_2 are subspaces of V.

Let $u, v \in W_1 \cap W_2$. Then $u, v \in W_1$ and also $u, v \in W_2$. Since W_1 is a subspace, it is closed under addition and $u+v \in W_1$. The same is true for W_2, so $u+v \in W_2$ and hence $u+v \in W_1 \cap W_2$. Suppose $\lambda \in \mathbb{F}$. Again, $\lambda v \in W_1$ and $\lambda v \in W_2$ since W_1 and W_2 are closed under scalar multiplication. So $\lambda v \in W_1 \cap W_2$. Finally, $0 \in W_1$ and $0 \in W_2$ since all subspaces of V contain $0 \in V$, so $0 \in W_1 \cap W_2$. □

(b) **False.**

For example, take $V = W_2$ and $W_1 = \{0\}$. Then in particular, $0 \notin W_2 - W_1$ so $W_2 - W_1$ cannot be a subspace.

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