Math 115A - Spring 2019
Exam 2

Full Name: ______________________________

UID: ______________________________

Instructions:

- Read each problem carefully.
- Show all work clearly and circle or box your final answer where appropriate.
- Justify your answers. A correct final answer without valid reasoning will not receive credit.
- All work including proofs should be well organized and clearly written using complete sentences.
- You may use the provided scratch paper, however this work will not be graded unless very clearly indicated there and in the exam.
- Calculators are not allowed but you may have a 3×5 inch notecard.

Page	Points	Score
1	10	
2	10	
3	10	
4	10	
Total:	40	

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You may use this page for scratch work. Work found on this page will not be graded unless clearly indicated here and in the exam.

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1. (10 points) True or False: Prove or disprove the following statements.

   (a) Let $T : V \to V$ be a linear operator on a finite-dimensional vector space over a field $\mathbb{F}$. Let v and w be two eigenvectors of T with eigenvalue $\lambda \in \mathbb{F}$. Then any nonzero linear combination of v and w is also an eigenvector of T.

   (b) Let $S, T : V \to V$ be linear operators on a finite-dimensional vector space. Assume that S and T commute, i.e. that $ST = TS$. If T is injective then S is injective.

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2. (10 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by reflection about the line $y = 2x$.

   (a) Give a basis for $\mathbb{R}^2$ consisting of eigenvectors for T and find their corresponding eigenvalues.

   (b) Is there a basis γ for $\mathbb{R}^2$ such that $[T]_\gamma^\gamma$ is the following matrix?

$$[T]_\gamma^\gamma = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}$$

   If so, find the basis γ. If not, justify why no such basis exists.

3. (10 points) Let $A, B \in M_{n\times n}(\mathbb{F})$ and let $\text{tr}(A) = \sum_{i=1}^{n} A_{ii}$ be the trace of A.

(a) Show that if A and B are similar then $\text{tr}(A) = \text{tr}(B)$.

(b) Show that if $A^k = 0$ for some $k \geq 1$ then the determinant $\det(A) = 0$.

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4. (10 points) Let $V = M_{2\times 2}(\mathbb{R})$ and $W = P_3(\mathbb{R})$. Let

$$\beta = \left\{ w_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, w_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, w_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, w_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \text{ and}$$
$$\gamma = \{1, x, x^2, x^3\}$$

be the standard bases. Consider the linear map $T : V \to W$ defined by

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a-c)x^3 + (a+c-2b+2d)x^2 + 3(c+d)x + 2(c+d).$$

(a) Find $[T]_\beta^\gamma$.

(b) Prove that although $V \cong W$, the map T is not an isomorphism. (*Hint:* The proof that $V \cong W$ should be one line.)

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