# Math 115A - Spring 2019
# Practice Exam 2 - Solutions

**Full Name:** ______________________________

**UID:** ______________________________

**Instructions:**

- Read each problem carefully.
- Show all work clearly and circle or box your final answer where appropriate.
- Justify your answers. A correct final answer without valid reasoning will not receive credit.
- All work including proofs should be well organized and clearly written using complete sentences.
- You may use the provided scratch paper, however this work will not be graded unless very clearly indicated there and in the exam.
- Calculators are not allowed but you may have a $3 \times 5$ inch notecard.

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| Page | Points | Score |
|---|---|---|
| 1 | 10 | |
| 2 | 10 | |
| 3 | 15 | |
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| Total: | 50 | |

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You may use this page for scratch work. Work found on this page will not be graded unless clearly indicated here and in the exam.

1. (10 points) True or False: Prove or disprove the following statements.

   (a) If $T : V \to W$ is a linear map between two $n$-dimensional vector spaces then $T$ is onto if and only if $T$ is one-to-one.

   (b) If $T : V \to W$ is a linear map between two finite-dimensional vector spaces then $T$ is an isomorphism if and only if $T$ maps any basis $\beta$ for $V$ to a basis $T(\beta)$ for $W$.

**Solution:**

(a) **True.**

*Proof.* ($\Longrightarrow$) If $T$ is onto then $\operatorname{im} T = W$ so $\operatorname{rank} T = \dim W = n$. By the dimension theorem (or rank-nullity),

$$n = \dim V = \operatorname{rank} T + \operatorname{null} T.$$

Then we calculate

$$\dim(\ker T) = \operatorname{null} T = n - \operatorname{rank} T = n - n = 0$$

and so it must be that $\ker T = \{0\}$. Thus $T$ is one-to-one.

($\Longleftarrow$) If $T$ is one-to-one, then $\ker T = \{0\}$ and so $\operatorname{null} T = 0$. Again by the dimension theorem

$$\dim(\operatorname{im} T) = \operatorname{rank} T = \dim V - \operatorname{null} T = n - 0 = n = \dim W$$

so $T$ is onto. □

(b) **True.**

*Proof.* ($\Longrightarrow$) Suppose $T$ is an isomorphism. If $\beta = \{v_1, \ldots, v_n\}$ is a basis for $V$ then $\operatorname{im} T = \operatorname{span} T(\beta) = \operatorname{span}\{T(v_1), \ldots, T(v_n)\}$. This follows because clearly $T(\beta) \subseteq \operatorname{im} T$ and so $\operatorname{span} T(\beta) \subseteq \operatorname{im} T$. Furthermore, if $w \in \operatorname{im} T$ then there exists $v \in V$ such that $T(v) = w$. Writing $v$ as a linear combination of the vectors in $\beta$ and applying the linear map $T$ gives $w$ as a linear combination of the vectors in $T(\beta)$, so $\operatorname{im} T \subseteq \operatorname{span} T(\beta)$.

Now since $T$ is an isomorphism, $T$ is onto and $\operatorname{im} T = W$. This means $T(\beta)$ spans $W$. But by the classification of finite-dimensional vector spaces $V \cong W$ if and only if $\dim V = \dim W$. Since $\beta$ is a basis for $V$, it must be that $n = \dim V = \dim W$. Because $T(\beta) = \{T(v_1), \ldots, T(v_n)\}$ spans $W$ and contains $n$ vectors, it must be a basis for $W$.

($\Longleftarrow$) Now suppose $T$ maps any basis $\beta$ for $V$ to a basis $T(\beta)$ for $W$. Then $\dim V = \dim W$ since $\beta$ and $T(\beta)$ have the same number of elements. We see that $T$ is onto since we showed above $\operatorname{im} T = \operatorname{span} T(\beta)$ and $T(\beta)$ is a basis for $W$. Finally, by part (a) we know that $T$ is also one-to-one and hence an isomorphism. □

2. (10 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the projection onto the $x$-axis along the line $y = 2x$.

   (a) Give a basis for $\mathbb{R}^2$ consisting of eigenvectors for $T$ and find their corresponding eigenvalues.

   (b) Find the matrix $T$ in the standard basis for $\mathbb{R}^2$.

**Solution:**

(a) Since $T$ is projection onto the $x$-axis, any vector of the form $(x, 0)$ is fixed by $T$, i.e. $T(x, 0) = (x, 0)$. So in particular $(1, 0)$ is an eigenvector with eigenvalue $\lambda = 1$. We are projecting along the line $y = 2x$, so any vector along this line is sent to zero. In particular $T(1, 2) = 0(1, 2)$ so $(1, 2)$ is an eigenvector with eigenvalue $\lambda = 0$. Since $(1, 0)$ and $(2, 1)$ are linearly independent, we can take as a basis for $\mathbb{R}^2$ the eigenvectors $\boxed{\{(1, 0), (1, 2)\}}$. (Note: we can check directly that the two vectors are linearly independent, but we have also shown in class that eigenvectors corresponding to distinct eigenvalues are linearly independent).

(b) Let $\beta$ be the standard basis for $\mathbb{R}^2$ given by $\{e_1, e_2\}$. From part (a), we can compute that $T$ represented by a matrix in the basis $\beta'$ is diagonal and so

$$[T]_{\beta'}^{\beta'} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Now we find the change of basis matrix $Q = [I]_{\beta}^{\beta'}$ since then

$$[T]_{\beta}^{\beta} = Q^{-1}[T]_{\beta'}^{\beta'}Q.$$

In this instance, it is easier to compute $Q^{-1} = [I]_{\beta'}^{\beta}$ as it has columns given by the vectors in $\beta'$ so

$$Q^{-1} = [I]_{\beta'}^{\beta} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}.$$

Then we compute

$$Q = \begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}.$$

So finally we have

$$[T]_{\beta}^{\beta} = Q^{-1}[T]_{\beta'}^{\beta'}Q = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{pmatrix}.$$

Thus $\boxed{[T]_{\beta}^{\beta} = \begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{pmatrix}}$.

3. (15 points) Let $\beta = \{1, x, x^2\}$ and $\beta' = \{1 + x + x^2, x + x^2, x^2\}$ be bases of $P_2(\mathbb{R})$.

(a) Find the change of coordinate matrix from $\beta'$ to $\beta$.

(b) Find the characteristic polynomial for the matrix found in part (a).

(c) Find the change of coordinate matrix from $\beta$ to $\beta'$.

**Solution:**

(a) We compute the change of basis matrix $[I]_{\beta'}^{\beta}$ as

$$\boxed{[I]_{\beta'}^{\beta} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}}.$$

(b) This is not a very well posed question as we should only find the characteristic polynomial for a matrix of the form $[T]_{\beta}^{\beta}$. However, we can call the matrix we found above $A$ and compute the charatristic polynomial as $p_A(t) = \det(A - tI)$. In that case we have

$$p_A(t) = \det(A - tI) = \det \begin{pmatrix} 1-t & 0 & 0 \\ 1 & 1-t & 0 \\ 1 & 1 & 1-t \end{pmatrix} = (1-t)^3.$$

So $\boxed{p_A(t) = (1-t)^3}$.

(c) To find the change of basis matrix $[I]_{\beta}^{\beta'}$, we can either write each element of the standard basis $\beta$ in terms of $\beta'$ or find the inverse of the matrix in part (a). In either case, we should have

$$\boxed{[I]_{\beta}^{\beta'} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}}.$$



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4. (15 points) Let $V = P_3(\mathbb{R})$ and $W = M_{2\times 2}(\mathbb{R})$. Let

$$\beta = \{1, x, x^2, x^3\}$$

$$\gamma = \left\{ w_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, w_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, w_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, w_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

be the standard bases. Consider the linear map $T : V \to W$ defined by

$$T(ax^3 + bx^2 + cx + d) = \begin{pmatrix} a+b & c+d \\ a+c & b+c \end{pmatrix}.$$

(a) Determine $M = [T]_\beta^\gamma$.

(b) Prove that $T$ is an isomorphism.

(c) Prove that $V$ and $W$ are isomorphic without using $T$.

**Solution:**

(a) We need to express $T(1), T(x), T(x^2), T(x^3)$ in the $\gamma$ basis. So we compute

$$T(1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = w_2$$
$$T(x) = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = w_2 + w_3 + w_4$$
$$T(x^2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = w_1 + w_4$$
$$T(x^3) = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = w_1 + w_3.$$

Collecting up the coefficients we have

$$\boxed{[T]_\beta^\gamma = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}}.$$

(b) *Proof.* We know that $T$ is an isomorphism if and only if $T$ is invertible. But $T$ is invertible if and only if every matrix representation of $T$ is invertible. We can compute that $\det[T]_\beta^\gamma = -2 \neq 0$ so $T$ is invertible.

Alternatively, $T$ is a linear map between two four-dimensional vector spaces. If $T$ is one-to-one then $T$ is an isomorphism. So we can compute the kernel

$$\ker T = \left\{ (ax^3 + bx^2 + cx + d) \;\middle|\; \begin{pmatrix} a+b & c+d \\ a+c & b+c \end{pmatrix} = 0 \right\}.$$



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We get a system of equations

$$\begin{cases} a+b=0 \\ c+d=0 \\ a+c=0 \\ b+c=0 \end{cases}$$

where the first and third equations give $b = c$, but the last gives $b = -c$. Since we are working over the field $\mathbb{R}$, it must be that $b = c = 0$. But then also $a = d = 0$. So $\ker T = \{0\}$ and $T$ is indeed one-to-one. Thus $T$ is an isomorphism. $\square$

(c) *Proof.* Notice that $V$ and $W$ are both four-dimensional vector spaces. By the classification of finite-dimensional vector spaces $V \cong W$. $\square$



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