Math 115A - Spring 2019
Practice Final Exam - Solutions

Full Name: ______________________________

UID: ______________________________

Instructions:

- Read each problem carefully.
- Show all work clearly and circle or box your final answer where appropriate.
- Justify your answers. A correct final answer without valid reasoning will not receive credit.
- All work including proofs should be well organized and clearly written using complete sentences.
- You may use the provided scratch paper, however this work will not be graded unless very clearly indicated there and in the exam.
- Calculators are not allowed but you may have a 3×5 inch notecard.

Page	Points	Score
1	15	
2	10	
3	15	
4	15	
6	10	

Page	Points	Score
7	15	
8	10	
9	10	
Bonus		
Total:	100	

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1. (15 points) Consider the vector space $V = P_2(\mathbb{R})$ with standard basis

$$\beta = \{1, x, x^2\}$$

and the linear maps

$$T : V \to V, \quad T(f) = f(1) + f(-1)x + f(0)x^2,$$
$$S : V \to V, \quad S(ax^2 + bx + c) = cx^2 + bx + a.$$

(a) Find $[T]_\beta^\beta$ and $[S]_\beta^\beta$. Then show that

$$[TS]_\beta^\beta = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

(b) Compute $[(TS)^{-1}]_\beta^\beta$.

(c) What is $(TS)^{-1}(x^2 + x + 1)$?

Solution:

(a) We compute $T(1) = 1 + x + x^2$, $T(x) = 1 - x$, and $T(x^2) = 1 + x$. So

$$[T]_\beta^\beta = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

and similarly,

$$[S]_\beta^\beta = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Now multiplying the matrices

$$[TS]_\beta^\beta = [T]_\beta^\beta [S]_\beta^\beta = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

(b) We compute the inverse of $[TS]_\beta^\beta$ in the usual way

$$[(TS)^{-1}]_\beta^\beta = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -1 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then to find $(TS)^{-1}(x^2 + x + 1)$ we compute

$$[(TS)^{-1}(x^2 + x + 1)]_\beta = [(TS)^{-1}]_\beta^\beta [x^2 + x + 1]_\beta$$
$$= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -1 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

So $(TS)^{-1}(x^2 + x + 1) = x^2$.

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2. (10 points) Consider the matrix

$$A = \begin{pmatrix} 0 & 1 & -2 \\ 1 & 0 & -2 \\ 0 & 0 & -1 \end{pmatrix}$$

in $M_{3\times 3}(\mathbb{R})$.

(a) Compute the characteristic polynomial of A. Find all the eigenvalues of A and their algebraic multiplicities.

(b) Is A diagonalizable? If so, find a basis β of eigenvectors for A and write $[T_A]_\beta^\beta$.

Solution:

(a) We compute

$$p_A(t) = \det(A - tI) = \det \begin{pmatrix} -t & 1 & -2 \\ 1 & -t & -2 \\ 0 & 0 & -1-t \end{pmatrix} = (t^2-1)(-1-t) = -(t+1)^2(t-1).$$

The roots of the characteristic polynomial are $\lambda_1 = 1$ and $\lambda_2 = -1$ with algebraic multiplicities 1 and 2 respectively.

(b) We compute the geometric multiplicities of the eigenvalues, i.e. the dimensions of the eigenspaces. Solving

$$\begin{pmatrix} 0 & 1 & -2 \\ 1 & 0 & -2 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

we get the system of equations $\{y - 2z = x, x - 2z = y, -z = z\}$. We find a basis for E_1 is given by $\{(1,1,0)\}$. Similarly, for eigenvectors with eigenevalue -1 we get the system of equations $\{y - 2z = -x, x - 2z = -y, -z = -z\}$. So a basis for E_{-1} is given by $\{(2,0,1),(0,2,1)\}$.

Thus the geometric multiplicities agree with the algebraic multiplicities and A is diagonalizable. Let $\beta = \{(1,1,0),(2,0,1),(0,2,1)\}$ be our basis of eigenvectors. Then we have

$$[T_A]_\beta^\beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

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3. (15 points) Consider the vector space $V = \mathbb{R}^4$ with the standard inner product. Let S be

$$S = \{w_1 = (1,0,1,0), w_2 = (1,1,1,1), w_3 = (2,2,0,2)\}.$$

(a) Apply the Gram-Schmidt orthogonalization algorithm to S to compute an orthogonal basis β' of span(S). You may use that S is linearly independent.

(b) Use your result from part (a) to compute an orthonormal basis β of span(S).

(c) Let $x = (1,2,3,2) \in$ span(S). Compute the coordinate vector $[x]_\beta$.

Solution:

(a) In the Gram-Schmidt algorithm we set

$$v_1 = w_1 \text{ and } v_k = w_k - \sum_{j=1}^{k-1} \frac{\langle w_k, v_j \rangle}{||v_j||^2} v_j.$$

So we compute

$$\begin{aligned} v_1 &= w_1 = (1,0,1,0) \\ v_2 &= w_2 - \frac{\langle w_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 = (1,1,1,1) - \frac{2}{2}(1,0,1,0) = (0,1,0,1) \\ v_3 &= w_3 - \frac{\langle w_3, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 - \frac{\langle w_3, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2 \\ &= (2,2,0,2) - \frac{2}{2}(1,0,1,0) - \frac{4}{2}(0,1,0,1) \\ &= (1,0,-1,0). \end{aligned}$$

So an orthogonal basis β' for span(S) is given by

$$\beta' = \{v_1 = (1,0,1,0), v_2 = (0,1,0,1), v_3 = (1,0,-1,0)\}.$$

(b) Now an orthonormal basis β for span(S) is given by

$$\beta = \left\{ u_1 = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0\right), u_2 = \left(0, \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), u_3 = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}, 0\right) \right\}.$$

(c) To find $[x]_\beta$ we use that $x = \sum_{i=1}^{3} \langle x, u_i \rangle u_i$. So we compute

$$x = \sum_{i=1}^{3} \langle x, u_i \rangle u_i = 2\sqrt{2}u_1 + 2\sqrt{2}u_2 - \sqrt{2}u_3.$$

Hence $[x]_\beta = \left(2\sqrt{2}, 2\sqrt{2}, -\sqrt{2}\right)$.

4. (15 points) Let V be a finite-dimensional vector space over $\mathbb{R}$ with an inner product so that $\langle x, y\rangle \in \mathbb{R}$ for $x, y \in V$.

(a) Let $\lambda \in \mathbb{R}$ with $\lambda > 0$. Show that

$$\langle x, y\rangle' = \lambda \langle x, y\rangle$$

for $x, y \in V$ defines an inner product on V.

(b) The inner product on V defines an induced norm. Show that

$$\langle x, y\rangle = \frac{1}{2}\left(||x+y||^2 - ||x||^2 - ||y||^2\right)$$

for all $x, y \in V$. Hence the inner product can be recovered from the norm.

(c) Let $\beta = \{v_1, \ldots, v_n\}$ be a basis for V. The *Gram matrix* $G \in M_{n\times n}(\mathbb{R})$ of the inner product $\langle -, -\rangle$ with respect to the basis β is defined by

$$G_{ij} = \langle v_i, v_j\rangle .$$

Show that G is invertible.

Solution:

(a) *Proof.* We need to check the four properties of an inner product. Let $u, v, w \in V$ and $c \in \mathbb{R}$. The properties all follow from the fact that $\langle -, -\rangle$ is an inner product. For linearity in the first coordinate,

$$\langle u+v, w\rangle' = \lambda \langle u+v, w\rangle = \lambda \langle u, w\rangle + \lambda \langle v, w\rangle = \langle u, w\rangle' + \langle v, w\rangle' .$$

Similarly,

$$\langle cv, w\rangle' = \lambda \langle cv, w\rangle = c\lambda \langle v, w\rangle = c \langle v, w\rangle'$$

and

$$\overline{\langle v, w\rangle'} = \overline{\lambda \langle v, w\rangle} = \overline{\lambda} \langle w, v\rangle = \lambda \langle w, v\rangle = \langle w, v\rangle' .$$

Finally we check that $\langle v, v\rangle' > 0$ if $v \neq 0$ and indeed

$$\langle v, v\rangle' = \lambda \langle v, v\rangle > 0$$

because $\lambda > 0$. □

(b) *Proof.* The norm of a vector is defined by $||x|| = \sqrt{\langle x, x\rangle}$. We begin by computing $||x+y||^2 = \langle x+y, x+y\rangle$. Then using the properties of an inner product we have

$$\begin{aligned} ||x+y||^2 &= \langle x+y, x+y\rangle \\ &= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle \\ &= ||x||^2 + \langle x, y\rangle + \overline{\langle x, y\rangle} + ||y||^2 \\ &= ||x||^2 + 2\langle x, y\rangle + ||y||^2 \end{aligned}$$

because V is a vector space over $\mathbb{R}$. Now solving for $\langle x, y\rangle$ we see that

$$\langle x, y\rangle = \frac{1}{2}\left(||x+y||^2 - ||x||^2 - ||y||^2\right)$$

as desired. □

(c) *Proof.* The matrix G defines a linear operator $T_G : V \to V$. We will show the kernel of this linear operator is $\{0\}$, which implies T_G is one-to-one. Then by the rank-nullity theorem $\text{rank}(T_G) = n = \dim V$ so T_G is also surjective and hence an isomorphism.

Let $x \in \ker T_G$. Then $0 = T_G(x) = G[x]_\beta$. Expanding x in terms of the basis β, we can write $x = \lambda_1 v_1 + \cdots + \lambda_n v_n$ for some scalars $\lambda_i \in \mathbb{R}$ for $1 \leq i \leq n$. Then $[x]_\beta = (\lambda_1, \ldots, \lambda_n)$ so

$$0 = G[x]_\beta = \begin{pmatrix} \langle v_1, v_1\rangle \lambda_1 + \cdots + \langle v_1, v_n\rangle \lambda_n \\ \vdots \\ \langle v_n, v_1\rangle \lambda_1 + \cdots + \langle v_n, v_n\rangle \lambda_n \end{pmatrix} = \begin{pmatrix} \langle v_1, x\rangle \\ \vdots \\ \langle v_n, x\rangle \end{pmatrix}.$$

This implies $x \in V^\perp$. But $V^\perp = \{0\}$ so $x = 0$ and we are done. □

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5. (10 points) Let V be a finite-dimensional vector space over a field $\mathbb{F}$ and let $S, T : V \to V$ be two linear operators.

   (a) Show that $\operatorname{rank}(ST) \leq \min\{\operatorname{rank}(S), \operatorname{rank}(T)\}$.

   (b) Suppose $T^2 = T$. Show that $\ker(T) \cap \operatorname{im}(T) = \{0\}$.

Solution:

(a) *Proof.* Let $z \in \operatorname{im}(ST)$. Then $z = ST(v)$ for some $v \in V$ and if $w = T(v)$ then $z = S(w)$. So $\operatorname{im}(ST) \subseteq \operatorname{im}(S)$ and hence $\operatorname{rank}(ST) \leq \operatorname{rank}(S)$.

It remains to show that we also have $\operatorname{rank}(ST) \leq \operatorname{rank}(T)$. Suppose $x \in \ker(T)$. Then $ST(x) = S(T(x)) = S(0) = 0$ so $\ker(T) \subseteq \ker(ST)$. This shows that $\operatorname{null}(T) \leq \operatorname{null}(ST)$. If $\dim V = n$ then by rank-nullity

$$n = \operatorname{rank}(ST) + \operatorname{null}(ST).$$

But then

$$\operatorname{null}(T) \leq \operatorname{null}(ST) = n - \operatorname{rank}(ST).$$

Applying rank-nullity for T we have

$$n - \operatorname{rank}(T) \leq n - \operatorname{rank}(ST)$$

and so $\operatorname{rank}(ST) \leq \operatorname{rank}(T)$ and we are done. □

(b) *Proof.* Notice that $0 \in \ker(T)$ and $0 \in \operatorname{im}(T)$ since both are subspaces of V. So we need to show 0 is the only element of $\ker(T) \cap \operatorname{im}(T)$. Assume for contradiction there exists $0 \neq v \in \ker(T) \cap \operatorname{im}(T)$. Then $T(v) = 0$ and there exists $w \in V$ such that $T(w) = v$. But since $T^2 = T$ we have

$$v = T(w) = T^2(w) = T(T(w)) = T(v) = 0,$$

a contradiction. Thus $\ker(T) \cap \operatorname{im}(T) = \{0\}$. □

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6. (15 points) True or False: Prove or disprove the following statements.

(a) An upper-triangular matrix is invertible if and only if all of its diagonal entries are nonzero.

(b) If $T : V \to V$ is an invertible linear operator then T is diagonalizable.

(c) If $T : V \to V$ is a diagonalizable linear operator then T is invertible.

Solution:

(a) **True.**

Proof. The determinant of an upper-triangular matrix is the product of the diagonal entries. This product will be nonzero precisely when all of the diagonal entries are nonzero. □

(b) **False.** Let $V = \mathbb{R}^2$ and consider the matrix

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The determinant of A is $1 \neq 0$ so A is invertible. The characteristic polynomial of A is

$$p_A(t) = \det(A - tI) = \det \begin{pmatrix} -t & 1 \\ -1 & -t \end{pmatrix} = t^2 + 1,$$

which has complex roots $i, -i$. The characteristic polynomial does not split over $\mathbb{R}$ and so A is not diagonalizable.

(c) **False.** Again consider $V = \mathbb{R}^2$ and let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Clearly A is already diagonal but $\det A = 0$ so A is not invertible.

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7. (10 points) Consider $\mathbb{C}$ as a vector space over $\mathbb{R}$ and define $\langle -, - \rangle : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ via

$$\langle w, z \rangle = \frac{1}{2} \left(w\overline{z} + z\overline{w} \right)$$

for all $w, z \in \mathbb{C}$.

(a) Show that $\langle -, - \rangle$ defined above is an inner product on $\mathbb{C}$.

(b) Let $T : \mathbb{C} \to \mathbb{C}$ be defined by $T(z) = \overline{z}$. Show that T is an isometry.

Solution:

(a) *Proof.* We check the four properties of an inner product hold. Let $v, w, z \in \mathbb{C}$ and $\lambda \in \mathbb{R}$. Then

$$\langle v + w, z \rangle = \frac{1}{2} \left((v + w)\overline{z} + z\overline{(v + w)} \right) = \frac{1}{2} \left(v\overline{z} + z\overline{v} + w\overline{z} + z\overline{w} \right) = \langle v, z \rangle + \langle w, z \rangle$$

so linearity in the first coordinate holds. Since $\lambda \in \mathbb{R}$ we also have

$$\langle \lambda w, z \rangle = \frac{1}{2}(\lambda w\overline{z} + z\overline{\lambda w}) = \lambda \frac{1}{2} \left(w\overline{z} + z\overline{w} \right) = \lambda \langle w, z \rangle .$$

We also need to check that conjugating the inner product behaves properly, and indeed,

$$\overline{w, z} = \overline{\frac{1}{2} \left(w\overline{z} + z\overline{w} \right)} = \frac{1}{2} \left(\overline{w}z + w\overline{z} \right) = \langle z, w \rangle .$$

Finally, if $z \neq 0$ then

$$\langle z, z \rangle = \frac{1}{2} \left(z\overline{z} + z\overline{z} \right) = \frac{1}{2} \left(|z|^2 + |z|^2 \right) = |z|^2 > 0.$$

Thus this does define an inner product on $\mathbb{C}$ as a real vector space. □

(b) *Proof.* We need to show that $\langle Tw, Tz \rangle = \langle w, z \rangle$ for all $w, z \in \mathbb{C}$. This follows easily since

$$\langle Tw, Tz \rangle = \frac{1}{2} \left(Tw\overline{Tz} + Tz\overline{Tw} \right) = \frac{1}{2} \left(\overline{w}z + \overline{z}w \right) = \langle w, z \rangle .$$

Thus T is an isometry. □

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8. (10 points) True or False: Prove or disprove the following statements.

Let V be a finite-dimensional inner product space over $\mathbb{F} = \mathbb{C}$. Let $T : V \to V$ be a a linear operator and T^* its adjoint.

(a) The linear operator $S = T + T^*$ is diagonalizable.

(b) If T is normal then $||Tv|| = ||T^*v||$ for all $v \in V$.

Solution:

(a) **True.**

Proof. Since $S = T + T^*$ and $T^{**} = T$, we see that $S^* = T^* + T = S$. So $S^*S = S^2 = SS^*$ and S is both self-adjoint and normal. Then by the spectral theorem for normal operators, since V is a complex vector space, there exists an orthonormal basis of eigenvectors for S. Hence S is diagonalizable. □

(b) **True.**

Proof. Since T is normal $T^*T = TT^*$. Then for any $v \in V$ we compute

$$\begin{aligned} ||Tv||^2 &= \langle Tv, Tv \rangle \\ &= \langle v, T^*Tv \rangle \\ &= \langle v, TT^*v \rangle \\ &= \langle T^*v, T^*v \rangle \\ &= ||T^*v||^2. \end{aligned}$$

So indeed $||Tv|| = ||T^*v||$. □

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