Math 115B - Winter 2020
Final Exam

Full Name: ______________________________

UID: ______________________________

I affirm that the work presented here is my own and that I have not given nor received any unauthorized assistance on this exam.

Signature: ______________________________

Instructions:

- Read each problem carefully.
- Show all work clearly and circle or box your final answer where appropriate.
- Justify your answers. A correct final answer without valid reasoning will not receive credit.
- All work including proofs should be well organized and clearly written using complete sentences.
- If a statement is true, provide a proof. If a statement is false, provide a counterexample demonstrating why it is false.
- You may use your notes, the textbook, reading assignments, and homework.
- You may not consult any outside resources including other people, the internet, or other textbooks.
- When you have finished the exam, upload your solutions as a PDF to Gradescope with at most one problem per page and each problem number clearly indicated.
- When uploading, include a cover page with your full name, UID, and the honor statement above.
- You may write your solutions on blank paper or directly on the exam. Your solutions may be written on paper, on a tablet, or using tex, as long as you upload your solutions as a PDF.

1. (5 points) True or False: Prove or disprove the following statement.

   Let $T\colon V \to V$ be a linear operator on a vector space V over a field $\mathbb{F}$ and let $p(t)$ be a polynomial in $\mathbb{F}[t]$. If λ is an eigenvalue of T then $p(\lambda)$ is an eigenvalue of $p(T)$.

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2. (5 points) Let $S, T\colon V \to V$ be self-adjoint operators on a finite-dimensional complex inner product space that commute. Show there exists and orthonormal basis for V consisting entirely of eigenvectors for both S and T.

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3. (5 points) Let V be a finite-dimensional complex inner product space and let $T: V \to V$ be a self-adjoint linear operator. Suppose that T is postive semidefinite. Show there exists a linear operator $S: V \to V$ such that $T = S^*S$.

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4. (5 points) Let V be an inner product space and let $W \subseteq V$ be a subspace. Then W is itself an inner product space by simply restricting the inner product from V. Let $T\colon W \to V$ be the inclusion so that $T(w) = w$ for all $w \in W$. Let $P\colon V \to W$ be the orthogonal projection onto W. Recall the generalized definition for adjoints of linear maps, not just linear operators. Show that the adjoint $T^* = P$.

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5. (5 points) Let V be a finite-dimensional vector space and let $W \subseteq V$ be a subspace. Consider the annihilator $W^0 = \{f \in V^* \mid f(w) = 0 \text{ for all } w \in W\} \subseteq V^*$. Show that $\dim W + \dim W^0 = \dim V$.

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6. (5 points) Let V be a finite-dimensional vector space and let $T\colon V \to V$ be a projection so $T^2 = T$. Show that each eigenvalue of T is either 0 or 1 and use this to prove that T is diagonalizable.

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7. (5 points) Find the characteristic polynomial, the minimal polynomial, and the Jordan canonical form of the linear transformation $T\colon \mathbb{R}^4 \to \mathbb{R}^4$ with matrix in the standard basis β given by

$$[T]_\beta = \begin{pmatrix} 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 \end{pmatrix}.$$

Be sure to justify your answers completely.

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8. (5 points) True or False: Prove or disprove the following statement.

   There exists a matrix $A \in M_{n\times n}(\mathbb{R})$ such that $A^n \neq 0$ but $A^{n+1} = 0$.

   (Hint: To prove this statement it suffices to provide an example A and justify that it satisfies these properties. To disprove the statement requires showing no such matrix exists).

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9. (5 points) Let $V = M_{2\times 2}(\mathbb{R})$ and define the function $\langle -, - \rangle \colon V \times V \to \mathbb{R}$ by letting $\langle A, B\rangle = \operatorname{tr}(AB)$. Show that this defines a symmetric bilinear form on V and find the matrix representing this bilinear form with respect to the basis

$$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

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10. (5 points) Let $A \in M_{n\times n}(\mathbb{R})$ be a symmetric matrix such that $v^t A v \leq 0$ for all $v \in \mathbb{R}^n$. Show that if $\mathrm{tr}(A) = 0$, i.e. the trace of A vanishes, then $A = 0$.

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11. (5 points) Let $A \in M_{n\times n}(\mathbb{C})$ and consider the subspace $W \subseteq M_{n\times n}(\mathbb{C})$ given by

$$W = \operatorname{span}\{I, A, A^2, A^3, \dots\}.$$

Show that $\dim(W) \leq n$. *Note:* Since $\dim\left(M_{n\times n}(\mathbb{C})\right) = n^2$ we know $\dim(W) \leq n^2$. The statement here is that in fact, $\dim(W) \leq n$.

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12. (5 points) Let $T\colon V \to V$ be a linear operator on a seven-dimensional vector space V over $\mathbb{F} = \mathbb{R}$. Suppose T has characteristic polynomial

$$p_T(t) = (1-t)^2(2-t)^2(3-t)^3$$

and that

$$\dim\left(\ker(T-I)\right) = 2 \qquad \dim\left(\ker(T-2I)\right) = 1 \qquad \dim\left(\ker(T-3I)\right) = 1.$$

Find a matrix that gives the Jordan canonical form of T. (Really this is "a" Jordan canonical form, but it is unique up to permutation of Jordan blocks.) Be sure to justify your answer.

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