Math 115B - Winter 2020
Practice Midterm Exam - Solutions

Full Name: ________________________________

UID: ________________________________

Instructions:

- Read each problem carefully.
- Show all work clearly and circle or box your final answer where appropriate.
- Justify your answers. A correct final answer without valid reasoning will not receive credit.
- All work including proofs should be well organized and clearly written using complete sentences.
- You may use the provided scratch paper, however this work will not be graded unless very clearly indicated there and in the exam.
- Calculators are not allowed but you may have a 3×5 inch notecard.

Page	Points	Score
1	10	
2	10	
3	10	
4	10	
Total:	40	

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1. (10 points) True or False: Prove or disprove the following statements.

Let V be a finite-dimensional inner product space over $\mathbb{F} = \mathbb{C}$. Let $T : V \to V$ be a a linear operator and T^* its adjoint.

(a) The linear operator $S = T + T^*$ is diagonalizable.

(b) If T is normal then $||Tv|| = ||T^*v||$ for all $v \in V$.

Solution:

(a) **True.**

Proof. Since $S = T + T^*$ and $T^{**} = T$, we see that $S^* = T^* + T = S$. So $S^*S = S^2 = SS^*$ and S is both self-adjoint and normal. Then by the spectral theorem for normal operators, since V is a complex vector space, there exists an orthonormal basis of eigenvectors for S. Hence S is diagonalizable. □

(b) **True.**

Proof. Since T is normal $T^*T = TT^*$. Then for any $v \in V$ we compute

$$\begin{aligned} ||Tv||^2 &= \langle Tv, Tv \rangle \\ &= \langle v, T^*Tv \rangle \\ &= \langle v, TT^*v \rangle \\ &= \langle T^*v, T^*v \rangle \\ &= ||T^*v||^2. \end{aligned}$$

So indeed $||Tv|| = ||T^*v||$. □

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2. (10 points) Let V be a finite-dimensional vector space and let T and S be linear operators on V. Suppose V is a T-cyclic subspace of itself. Show that T and U commute if and only if $U = g(T)$ for some polynomial $g(t)$.

Solution:

Proof. ($\Longrightarrow$) Assume that $TU = UT$. Since V is finite-dimensional and a T-cyclic subspace of itself, there exists $v \in V$ with $\beta = \{v, T(v), T^2(v), \ldots, T^{n-1}(v)\}$ a basis for V. Since $U(v) \in V$, there exist scalars $a_0, \ldots, a_{n-1}$ such that

$$U(v) = a_0 v + a_1 T(v) + \cdots + a_{n-1} T^{n-1}(v).$$

Let $g(t) = a_0 + a_1 t + \cdots + a_{n-1} t^{n-1}$. Then $U(v) = g(T)(v)$. Furthermore, for $1 \leq k \leq n-1$, since T and U commute we have

$$\begin{aligned}(U - g(T))(T^k(v)) &= UT^k(v) - g(T)T^k(v) \\ &= T^k U(v) - T^k g(T)(v) \\ &= T^k(U(v) - g(T)(v)) \\ &= 0.\end{aligned}$$

Thus $U - g(T)$ is zero on every element of the basis for V and so $U - g(T) = 0$, which completes the proof that $U = g(T)$.

($\Longleftarrow$) Now assume that $U = g(T)$ for some polynomial $g(t)$. Then

$$TU = Tg(T) = g(T)T = UT$$

since T commutes with any power of itself. □

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3. (10 points) Let $T\colon V \to V$ be a linear operator on a finite-dimensional vector space over a field $\mathbb{F}$. Let $T^t\colon V^* \to V^*$ be its dual. Show that a subspace $W \subseteq V$ is T invariant if and only if W^0 is T^t-invariant.

Solution:

Proof. ($\Longrightarrow$) Assume that $W \subseteq V$ is T-invariant. Recall that

$$W^0 = \{f \in V^* \mid f(w) = 0 \text{ for all } w \in W\}$$

and that $T^t\colon V^* \to V^*$ is defined by $T^t(g) = g \circ T$ for all $g \in V^*$.

Let $f \in W^0$. We want to show $T^t(f) \in W^0$. That is, we want to show $T^t(f)(w) = 0$ for any $w \in W$. So let $w \in W$ and consider $T^t(f)(w) = f(T(w))$. Since W is T-invariant we have that $T(w) \in W$. Furthermore, since $f \in W^0$ it must be that $f(T(w)) = 0$. So indeed, W^0 is T^t-invariant.

($\Longleftarrow$) Now suppose W^0 is T^t-invariant. Let $w \in W$. We want to show $T(w) \in W$. Assume to the contrary that $T(w) \notin W$. Let $\{w_1, \ldots, w_k\}$ be a basis for W. Since $T(w) \notin W$ we can take the linearly independent set $\{w_1, \ldots, w_k, T(w)\}$ and extend it to a basis β for V. There exists f in the dual basis to β that evaluates to zero on each basis element of β except $f(T(w)) = 1$. Since $f(w_i) = 0$ for all i, the functional f is zero on all elements of W and by definition $f \in W^0$. But then $1 = f(T(w)) = T^t(f)(w)$ implying $T^t(f) \notin W^0$, contradicting that W^0 is T^t-invariant. Thus $T(w) \in W$ and W is T-invariant. $\square$

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4. (10 points) True or False: Prove or disprove the following statements.

(a) Let V be a finite-dimensional inner product space and let $T\colon V \to V$ be a linear operator. If all the eigenvalues of T are 1, then T must be an isometry.

(b) Let $\beta = \{1, x, x^2\}$ be the standard basis for $V = P_2(\mathbb{R})$. There exists a basis for V such that the dual basis for V^* is given by $\{f_0, f_1, f_2\}$ with $f_0(p(x)) = p(0)$, $f_1(p(x)) = p(1)$, and $f_2(p(x)) = p(2)$.

Solution:

(a) **False.** Consider $V = \mathbb{R}^2$ and let $T\colon V \to V$ be defined by $T(x, y) = (x, x + y)$. Then in the standard basis β we have

$$[T]_\beta^\beta = A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The characteristic polynomial is $p_T(t) = \det(T - tI) = \det(A - tI) = (1 - t)^2$. The only roots are 1 and thus the eigenvalues of T are all 1. However, $A \neq A^t$ so T is not orthogonal and hence not an isometry.

(b) **True.**

Proof. We can write any $p(x) = a_0 + a_1 x + a_2 x^2$. Then

$$\begin{aligned} f_0(p(x)) &= p(0) = a_0 \\ f_1(p(x)) &= p(1) = a_0 + a_1 + a_2 \\ f_2(p(x)) &= p(2) = a_0 + 2a_1 + 4a_2. \end{aligned}$$

In particular, $\{f_0, f_1, f_2\}$ is linearly independent so there exists a dual basis for V^{**} and since V^{**} is naturally isomorphic to V this corresponds to a basis for V.

Alternatively, we can construct the basis with this dual. After solving some systems of a equations or using Lagrange interpolation, let

$$\begin{aligned} p_0(x) &= 1 - \frac{3}{2}x + \frac{1}{2}x^2 \\ p_1(x) &= 2x - x^2 \\ p_2(x) &= -\frac{1}{2}x + \frac{1}{2}x^2. \end{aligned}$$

It remains to check that $\{p_0, p_1, p_2\}$ forms a basis and then verify that $\{f_0, f_1, f_2\}$ is its dual basis. Since $1 = p_0 + p_1 - p_2$, $x = p_1 + 2p_2$, and $x^2 = p_1 + 4p_2$, we see that $\{p_0, p_1, p_2\}$ forms a basis for V and we easily verify $f_i(p_j) = \delta_{ij}$. □

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