Math 32A - Winter 2019
Practice Exam 1

Full Name: Solutions

UID: ______________________

Circle the name of your TA and the day of your discussion:

Qi Gui   Talon Stark   Tianqi (Tim) Wu

Tuesday   Thursday

Instructions:

- Read each problem carefully.
- Show all work clearly and circle or box your final answer where appropriate.
- Justify your answers. A correct final answer without valid reasoning will not receive credit.
- Simplify your answers as much as possible.
- Include units with your answer where applicable.
- Calculators are not allowed but you may have a 3 × 5 inch notecard.

Page	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

1. (3 points) True or False? Given two vectors $\mathbf{a}$ and $\mathbf{b}$, if $||\mathbf{a}|| = ||\mathbf{b}||$ then $\mathbf{a} = \mathbf{b}$.

   (a) True. (b) False. A vector is determined by its magnitude and direction.

2. (3 points) True or False? For any two vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^3$, $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = 0$.

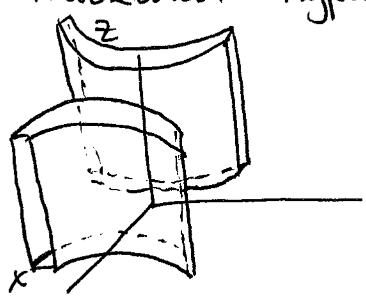
   (a) True. (b) False. $\vec{a} \times \vec{b}$ is orthogonal to $\vec{a}$ and to $\vec{b}$.

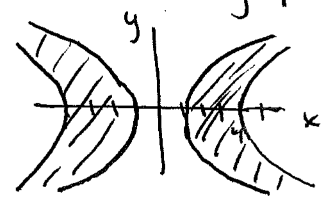
3. (4 points) Describe in words and sketch a picture of the region of $\mathbb{R}^3$ represented by the following inequality.

$$1 \leq \frac{x^2}{4} - y^2 \leq 4$$

trace in xy-plane

A thickened hyperbolic cylinder





4. (10 points) Let L be the line through the points $(1, 0, 7)$ [P] and $(3, -1, 5)$ [Q]. Does L intersect the plane $x + y + z = 6$? If not, justify your answer. If L does intersect the plane, find the intersection.

$L: \vec{r}(t) = \vec{r}_0 + t\vec{v}$

$\vec{v} = \vec{PQ} = \langle 3-1, -1-0, 5-7 \rangle = \langle 2, -1, -2 \rangle$

P is on L (or Q) so $\vec{r}_0 = \langle 1, 0, 7 \rangle$

$\vec{r}(t) = \langle 1, 0, 7 \rangle + t \langle 2, -1, -2 \rangle$

$\vec{r}(t) = \langle 1+2t, -t, 7-2t \rangle$

If L intersects plane: $x+y+z=6$

$(1+2t) + (-t) + (7-2t) = 6$

$8 - t = 6$

$2 = t$ $\rightsquigarrow$ intersects at a single point

$\vec{r}(2) = \langle 5, -2, 3 \rangle$ so intersection is $\boxed{(5, -2, 3)}$

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5. Consider the points $P = (3, 3, 1)$, $Q = (2, -1, 0)$, and $R = (-1, -3, 1)$.

(a) (10 points) Find the equation of the plane containing the P, Q, and R.

$\overrightarrow{PQ} = \langle 2-3, -1-3, 0-1 \rangle = \langle -1, -4, -1 \rangle$

$\overrightarrow{PR} = \langle -1-3, -3-3, 1-1 \rangle = \langle -4, -6, 0 \rangle$

$$\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & -4 & -1 \\ -4 & -6 & 0 \end{vmatrix} = \langle 0-6, -(0-4), 6-16 \rangle$$

$\vec{n} = \langle -6, 4, -10 \rangle \qquad P = (3, 3, 1)$

plane : $-6(x-3) + 4(y-3) - 10(z-1) = 0$

$\boxed{-6x + 4y - 10z = -16}$

(b) (5 points) Find the area of the triangle formed by the points P, Q, and R.

Area of parallelogram $= \|\overrightarrow{PQ} \times \overrightarrow{PR}\|$

Area of triangle $= \frac{1}{2}\|\overrightarrow{PQ} \times \overrightarrow{PR}\|$

$= \frac{1}{2}\|\langle -6, 4, -10 \rangle\| = \frac{1}{2}\sqrt{36 + 16 + 100}$

$= \frac{1}{2}\sqrt{152} = \frac{1}{2} 2\sqrt{38} = \boxed{\sqrt{38}}$

(or use $\langle -6, 4, -10 \rangle = 2\langle -3, 2, -5 \rangle$ so magnitude is $2\sqrt{9+4+25} = 2\sqrt{38}$)

(c) (5 points) Are the four points P, Q, R, and $S = (7, 4, 0)$ coplanar? Justify your answer.

coplanar if and only if scalar triple product $\overrightarrow{PS} \cdot (\overrightarrow{PQ} \times \overrightarrow{PR}) = 0$

$\overrightarrow{PS} = \langle 7-3, 4-3, 0-1 \rangle = \langle 4, 1, -1 \rangle$

$\overrightarrow{PS} \cdot (\overrightarrow{PQ} \times \overrightarrow{PR}) = \langle 4, 1, -1 \rangle \cdot \langle -6, 4, -10 \rangle = (4)(-6) + (1)(4) + (-1)(-10)$

$= -10 \neq 0$ so $\boxed{\text{no}}$

OR check whether S satisfies equation of the plane in part (a). $-6(7) + 4(4) - 10(0) \stackrel{?}{=} -16$

$-26 \neq -16$ so $\boxed{\text{no}}$

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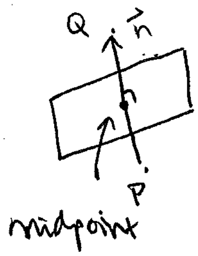
6. (5 points) Find the equation of the plane consisting of all points that are equidistant from the points $(3, 5, 6)$ and $(-5, 3, 2)$.



P Q

$\vec{n} = \overrightarrow{PQ} = \langle -5-3, 3-5, 2-6 \rangle = \langle -8, -2, -4 \rangle$

or take $\vec{n} = \overrightarrow{QP} = \langle 8, 2, 4 \rangle$

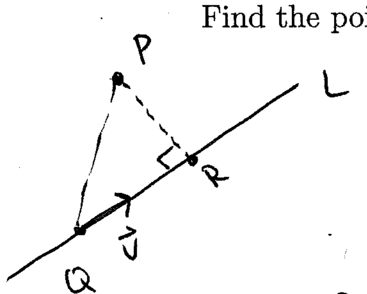
midpoint $\left(\frac{3+(-5)}{2}, \frac{5+3}{2}, \frac{6+2}{2} \right) = (-1, 4, 4)$

plane: $8(x+1) + 2(y-4) + 4(z-4) = 0$

$\boxed{8x + 2y + 4z = 16}$

7. (15 points) Let L be the line given parametrically by $x = 4 + t, y = -1 - 2t, z = 5 + t$. Find the point on the line L which is closest to the point $(-2, 2, -1)$.

P



$L: \begin{cases} x = 4 + t \\ y = -1 - 2t \\ z = 5 + t \end{cases}$

goes through $Q = (4, -1, 5)$ with direction vector $\vec{v} = \langle 1, -2, 1 \rangle$

$P = (-2, 2, -1)$ want R.

notice $\overrightarrow{QR} = \overrightarrow{QP}_{\parallel \vec{v}} = \text{proj}_{\vec{v}} \overrightarrow{QP}$ $\qquad \overrightarrow{QP} = \langle -6, 3, -6 \rangle$

$\vec{w}_{\parallel \vec{v}} = \text{proj}_{\vec{v}} \vec{w} = \left(\frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\|^2} \right) \vec{v}$ $\qquad$ so $\overrightarrow{QR} = \left(\frac{\overrightarrow{QP} \cdot \vec{v}}{\|\vec{v}\|^2} \right) \vec{v} = \frac{\langle -6, 3, -6 \rangle \cdot \langle 1, -2, 1 \rangle}{\|\langle 1, -2, 1 \rangle\|^2} \vec{v}$

$= \frac{-6-6-6}{1+4+1} \vec{v} = \frac{-18}{6} \vec{v} = -3\vec{v}$

$= \langle -3, 6, -3 \rangle$

So if $R = (a, b, c)$ $\qquad \overrightarrow{QR} = \langle a-4, b-(-1), c-5 \rangle = \langle -3, 6, -3 \rangle$

$a - 4 = -3, \quad b + 1 = 6, \quad c - 5 = -3$

$a = 1, \quad b = 5, \quad c = 2$

so $\boxed{R = (1, 5, 2)}$

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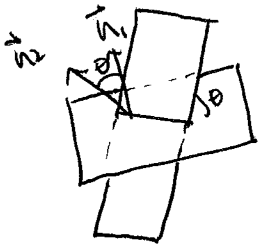
8. (10 points) Find the angle of intersection of the two planes $2x+y+z=6$ and $x+y-z=3$.



$$\vec{n}_1 = \langle 2,1,1 \rangle \qquad \vec{n}_2 = \langle 1,1,-1 \rangle$$

$$\vec{n}_1 \cdot \vec{n}_2 = \|\vec{n}_1\| \, \|\vec{n}_2\| \cos\theta$$

$$\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{\|\vec{n}_1\| \, \|\vec{n}_2\|} = \frac{\langle 2,1,1 \rangle \cdot \langle 1,1,-1 \rangle}{\sqrt{4+1+1}\,\sqrt{1+1+1}}$$

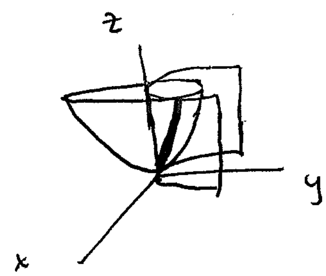
$$\cos\theta = \frac{2+1-1}{\sqrt{6}\sqrt{3}} = \frac{2}{\sqrt{18}} = \frac{2}{3\sqrt{2}}$$

$$\boxed{\theta = \cos^{-1}\left(\frac{2}{3\sqrt{2}}\right)}$$

9. (5 points) Find a vector function that represents the curve of intersection of the surfaces $z = 4x^2 + y^2$ and $y = x^2$.

$z = 4x^2 + y^2$ elliptic paraboloid

$y = x^2$ parabolic cylinder



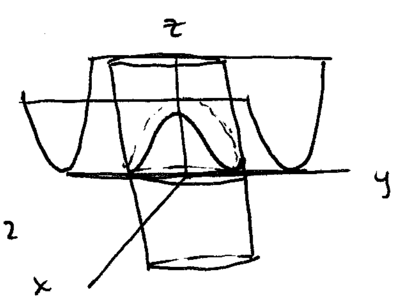
let $x = t \Rightarrow y = t^2 \quad z = 4t^2 + t^4$

$$\boxed{\vec{r}(t) = \langle t, t^2, 4t^2 + t^4 \rangle, \quad -\infty < t < \infty}$$

10. (5 points) Find a vector function that represents the curve of intersection of the surfaces $x^2 + y^2 = 4$ and $z = 4x^2$.

$x^2 + y^2 = 4$ circular cylinder

$z = 4x^2$ parabolic cylinder



Project to xy-plane $\rightarrow$ circle radius 2
center (0,0)

$x = 2\cos t, \ y = 2\sin t, \quad 0 \le t \le 2\pi \qquad$ then $z = 4x^2 = 4(2\cos t)^2 = 16\cos^2 t$

$$\boxed{\vec{r}(t) = \langle 2\cos t, 2\sin t, 16\cos^2 t \rangle, \ 0 \le t \le 2\pi}$$

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11. (5 points) Find a parametrization of the tangent line to $\mathbf{r}(t) = \langle 3t+2, t^2-7, t-t^2 \rangle$ at $\mathbf{r}(1)$.

$\vec{L}(t) = \vec{r}_0 + t\vec{r}\,'(t_0)$

$\vec{r}_0 = \vec{r}(1) = \langle 5, -6, 0 \rangle$

$\vec{r}\,'(t) = \langle 3, 2t, 1-2t \rangle$

$\vec{r}\,'(1) = \langle 3, 2, -1 \rangle$

$\vec{L}(t) = \langle 5, -6, 0 \rangle + t\langle 3, 2, -1 \rangle$

$\boxed{\vec{L}(t) = \langle 5+3t, -6+2t, -t \rangle}$

or $\boxed{\begin{cases} x = 5+3t \\ y = -6+2t \\ z = -t \end{cases}}$

12. (15 points) Find the angle between the following two curves at their point of intersection. The angle between the two curves is the angle between the tangent lines to the curves at their point of intersection.

$$\mathbf{r}(t) = \langle t^3, 2t^2+1, 2t+3 \rangle \quad \text{and} \quad \mathbf{p}(t) = \langle t-4, t-3, t-1 \rangle$$

First find the intersection: $\vec{r}(t) = \langle t^3, 2t^2+1, 2t+3 \rangle$

$\vec{p}(s) = \langle s-4, s-3, s-1 \rangle$

$\begin{cases} t^3 = s-4 \\ 2t^2+1 = s-3 \\ 2t+3 = s-1 \end{cases} \rightsquigarrow s = 2t+4 \rightsquigarrow 2t^2+1 = (2t+4)-3$

$2t^2-2t = 0$

$2t(t-1) = 0$

$t = 0, 1$

if $t=0$, $s=4$

~~if $t=1$, $s=6$~~

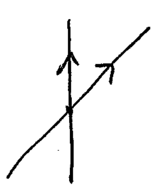
Check: $t^3 = s-4$

$t=0, s=4 \rightarrow 0=0$ ✓

$t=1, s=6$

$1 \neq 2$ X

So $t=0, s=4$



angle between tangent lines is the same as the angle between vectors parallel to these lines, e.g. $\vec{r}\,'$ and $\vec{p}\,'$

$\vec{r}\,'(t) = \langle 3t^2, 4t, 2 \rangle$

$\vec{r}\,'(0) = \langle 0, 0, 2 \rangle$

$\vec{p}\,'(s) = \langle 1, 1, 1 \rangle$ (b/c $\vec{p}$ is a line!)

$\vec{p}\,'(4) = \langle 1, 1, 1 \rangle$

$\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\|\vec{b}\|} = \dfrac{\langle 0,0,2 \rangle \cdot \langle 1,1,1 \rangle}{\sqrt{4}\sqrt{1+1+1}} = \dfrac{\cancel{2}}{\sqrt{\cancel{4}}\sqrt{3}} = \dfrac{1}{\sqrt{3}}$

so $\boxed{\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)}$

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