Math 32B - Fall 2019
Practice Final Exam

Full Name: Solutions

UID: ____________

Circle the name of your TA and the day of your discussion:

Steven Gagniere   Jason Snyder   Ryan Wilkinson

Tuesday   Thursday

Instructions:

- Read each problem carefully.
- Show all work clearly and circle or box your final answer where appropriate.
- Justify your answers. A correct final answer without valid reasoning will not receive credit.
- Simplify your answers as much as possible.
- Include units with your answer where applicable.
- Calculators are not allowed but you may have a 3×5 inch notecard.

Page	Points	Score	Page	Points	Score
1	10		6	15	
2	10		7	15	
3	10		8	15	
4	10		Bonus		
5	15		Total:	100	

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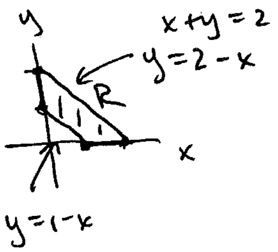
1. (5 points) Let $\mathcal{R}$ be the trapezoid with vertices $(1,0)$, $(2,0)$, $(0,2)$, and $(0,1)$. Evaluate the iterated integral $\iint_{\mathcal{R}} \cos\left(\frac{y-x}{y+x}\right)\, dA$.

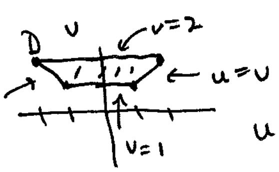


Both x and y simple but difficult integrand.

Let $u = y - x$, $v = y + x$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -2$$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{-2} = -\frac{1}{2}$$



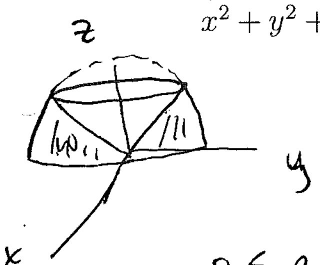
$$\text{So } \iint_R \cos\left(\frac{y-x}{y+x}\right) dA = \iint_D \cos\left(\frac{u}{v}\right)|J|\,du\,dv = \int_1^2 \int_{-v}^{v} \cos\left(\frac{u}{v}\right)\left|-\tfrac{1}{2}\right| du\,dv$$

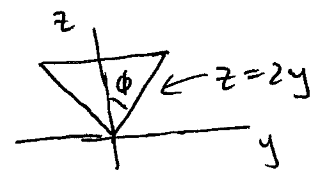
$$= \frac{1}{2}\int_1^2 \left[v \sin\left(\frac{u}{v}\right)\right]_{-v}^{v} dv = \frac{1}{2}\int_1^2 v\sin(1) - v\sin(-1)\,dv$$

odd

$$= \frac{1}{2}\int_1^2 v\sin(1) + v\sin(1)\,dv = \sin(1)\int_1^2 v\,dv = \sin(1)\left[\frac{v^2}{2}\right]_1^2 = \boxed{\frac{3}{2}\sin(1)}$$

2. (5 points) Find the volume of the solid that lies above the xy-plane, within the sphere $x^2 + y^2 + z^2 = 4$ and below the cone $z = 2\sqrt{x^2+y^2}$.





$\phi = \arctan\left(\frac{1}{2}\right)$

Spherical coordinates OR substitute

$z = 2\sqrt{x^2+y^2}$

in spherical coords.

$\rho\cos\phi = 2\rho\sin\phi$

$\frac{1}{2} = \tan\phi$

$0 \le \rho \le 2$

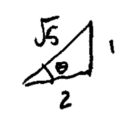
$0 \le \theta \le 2\pi$

$\arctan\left(\frac{1}{2}\right) \le \phi \le \frac{\pi}{2}$

$|J| = \rho^2 \sin\phi$

$$\text{So Vol} = \iiint_W 1\,dV = \int_{\arctan(\frac{1}{2})}^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^2 1 \cdot \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi$$

$\cos(\arctan(\frac{1}{2}))$



$= \frac{2}{\sqrt{5}}$

$$= \int_{\arctan(\frac{1}{2})}^{\frac{\pi}{2}} \sin\phi\,d\phi \int_0^{2\pi} d\theta \int_0^2 \rho^2\,d\rho$$

$$= \left[-\cos\phi\right]_{\arctan(\frac{1}{2})}^{\frac{\pi}{2}} \left[\theta\right]_0^{2\pi} \left[\frac{1}{3}\rho^3\right]_0^2 = \left(-0 + \frac{2}{\sqrt{5}}\right)(2\pi)\left(\frac{1}{3}8\right)$$

$$= \boxed{\frac{32\pi}{3\sqrt{5}}}$$

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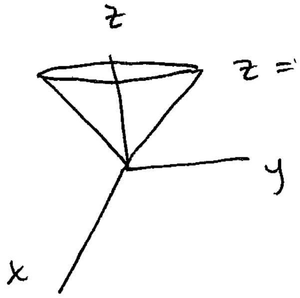
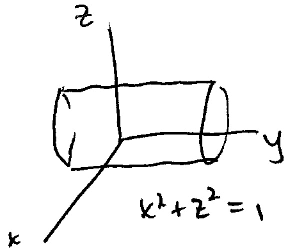
3. (10 points) Find the surface area of the portion $\mathcal{S}$ of $z = \sqrt{x^2+y^2}$ contained within the cylinder $x^2+z^2 \leq 1$.



$z = \sqrt{x^2+y^2}$

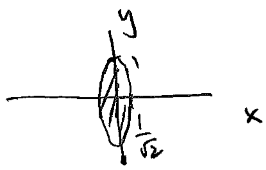


$SA = \iint\limits_{S} 1\, dS$

$z = \sqrt{x^2+y^2}$, $x^2+z^2=1$ $\rightsquigarrow$ $x^2+(x^2+y^2)=1$

$2x^2+y^2=1$

ellipse



$x = u$
$y = v$

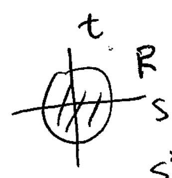
$\vec{r}(u,v) = \langle u, v, \sqrt{u^2+v^2} \rangle$, $\underset{D}{2u^2+v^2 \leq 1}$

$\vec{r}_u = \langle 1, 0, \frac{u}{\sqrt{u^2+v^2}} \rangle$

$\vec{r}_v = \langle 0, 1, \frac{v}{\sqrt{u^2+v^2}} \rangle$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & \frac{u}{\sqrt{u^2+v^2}} \\ 0 & 1 & \frac{v}{\sqrt{u^2+v^2}} \end{vmatrix} = \langle \frac{-u}{\sqrt{u^2+v^2}}, \frac{-v}{\sqrt{u^2+v^2}}, 1 \rangle$$

$$\|\vec{r}_u \times \vec{r}_v\| = \sqrt{\frac{u^2}{u^2+v^2} + \frac{v^2}{u^2+v^2} + 1} = \sqrt{\underbrace{\cancel{\frac{u^2+v^2}{u^2+v^2}}}_{1} + 1} = \sqrt{2}$$



$s^2+t^2 \leq 1$

$$SA = \iint\limits_{S} 1\, dS = \iint\limits_{D} \|\vec{r}_u \times \vec{r}_v\| \, du\,dv = \iint\limits_{D} \sqrt{2}\, du\,dv$$ change vars.

Let $s = \sqrt{2}u$, $t = v$ $\quad \frac{\partial(s,t)}{\partial(u,v)} = \begin{vmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{vmatrix} = \sqrt{2}$ so $\frac{\partial(u,v)}{\partial(s,t)} = \frac{1}{\sqrt{2}}$

$$\iint\limits_{D} \sqrt{2}\, du\,dv = \iint\limits_{R} \sqrt{2} \cdot \frac{1}{\sqrt{2}}\, ds\,dt = \iint\limits_{R} 1\, ds\,dt = A(R) = \boxed{\pi}$$

/10

4. (5 points) Evaluate the line integral $\int_C y \sin z \, ds$ where C is parameterized by $x = \cos t$, $y = \sin t$, $z = t$ for $0 \le t \le 2\pi$.

$\vec{r}(t) = \langle \cos t, \sin t, t \rangle, \quad 0 \le t \le 2\pi$ $\qquad \int_C f \, ds = \int_a^b f(\vec{r}(t)) \|\vec{r}\,'(t)\| \, dt$

$\vec{r}\,'(t) = \langle -\sin t, \cos t, 1 \rangle$

$\|\vec{r}\,'(t)\| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{1+1} = \sqrt{2}$

$$\int_C y \sin z \, ds = \int_0^{2\pi} \sin t \sin t \sqrt{2} \, dt = \int_0^{2\pi} \sqrt{2} \sin^2 t \, dt$$

$$= \int_0^{2\pi} \sqrt{2}\left(\tfrac{1}{2} - \tfrac{1}{2}\cos(2t)\right) dt = \frac{\sqrt{2}}{2} \int_0^{2\pi} 1 - \cos(2t) \, dt$$

$$= \frac{\sqrt{2}}{2} \left[t - \tfrac{1}{2}\sin(2t) \right]_0^{2\pi} = \frac{\sqrt{2}}{2} \cdot 2\pi = \boxed{\sqrt{2}\,\pi}$$

5. (5 points) Let $\mathbf{F}(x, y, z) = \langle 2xy^2 \cos z, 2x^2 y \cos z + 2y, -x^2y^2 \sin z + 1 \rangle$. Find the work done by the vector field $\mathbf{F}$ in moving a particle along the curve C parameterized by $\mathbf{r}(t) = \langle t, \sin t, t^2 + 1 \rangle$ for $0 \le t \le \pi$

$W = \int_C \vec{F} \cdot d\vec{r}$

$\vec{F}$ looks terrible but it is conservative!

$f(x,y,z) = x^2 y^2 \cos z + y^2 + z$

$\nabla f = \langle 2xy^2 \cos z, \ 2x^2 y \cos z + 2y, \ -x^2y^2 \sin z + 1 \rangle = \vec{F}$

So by the Fundamental Thm for Line Integrals

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} \overset{\text{Fund.}}{\underset{\text{Thm}}{=}} f(\vec{r}(\pi)) - f(\vec{r}(0))$$

$$= f(\pi, 0, \pi^2 + 1) - f(0, 0, 1)$$

$$= \pi^2 + 1 - 1 = \boxed{\pi^2}$$

/10

W

6. (5 points) Find the average value of $f(x, y, z) = xy^2z^3$ on the box $[0, 1] \times [0, 2] \times [0, 3]$.

$$f_{avg} = \frac{1}{vol(W)} \iiint_W f \, dV \qquad vol(W) = 1 \cdot 2 \cdot 3 = 6$$

$$f_{avg} = \frac{1}{6} \int_0^3 \int_0^2 \int_0^1 xy^2z^3 \, dx\,dy\,dz = \frac{1}{6} \int_0^3 z^3 dz \int_0^2 y^2 dy \int_0^1 x\,dx$$

$$= \frac{1}{6} \left(\frac{1}{4} z^4 \right]_0^3 \left[\frac{1}{3} y^3 \right]_0^2 \left[\frac{1}{2} x^2 \right]_0^1 = \frac{1}{6} \cdot \frac{1}{\cancel{4}} 3^{\cancel{4}\,3} \cdot \frac{1}{\cancel{3}} \cancel{2^3}^{\,4} \cdot \frac{1}{\cancel{2}} \cdot 1$$

$$= \frac{27}{6} = \boxed{\frac{9}{2}}$$

7. (5 points) The Laplace operator Δ of a function $f(x, y, z)$ is defined by

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

A function f satisfying $\Delta f = 0$ is called harmonic.

(a) Show that $\mathbf{F}(x, y, z) = \langle xz, -yz, \frac{1}{2}(x^2 - y^2) \rangle$ is the gradient of a harmonic function.

$$f(x,y,z) = \frac{1}{2}(x^2 - y^2) z \qquad \nabla f = \langle xz, -yz, \tfrac{1}{2}(x^2 - y^2) \rangle = \vec{F}.$$

$$\Delta f = \operatorname{div}(\nabla f) = \operatorname{div}(\vec{F}) = \nabla \cdot \vec{F} = \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(-yz) + \frac{\partial}{\partial z}\left(\tfrac{1}{2}(x^2 - y^2)\right)$$

$= z - z + 0 = 0$. So $\nabla f = \vec{F}$ and $\Delta f = 0$ ✓

(b) Find the flux of $\mathbf{F}$ through the surface S given by $x^2 + y^2 + z^2 = 1$ with outward normal.

$\vec{F}$ is ~~nice~~ differentiable on all of $\mathbb{R}^3$ – no holes

S – sphere is piecewise smooth, positively oriented

$S = \partial W$ where W = ball of radius 1

By the Divergence Thm flux across S

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_W \operatorname{div} \vec{F} \, dV = \iiint_W 0 \, dV = \boxed{0}$$

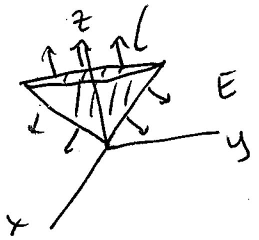
above

/10

8. (15 points) Let $\mathbf{F}(x,y,z) = \langle xy^2 + e^{y+\cos y}, x^2y + \sin z, z^2 + \cos x\rangle$ and let E be the solid cone consisting of the points above $z = \sqrt{x^2+y^2}$ and below $z = 4$. Find $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$ where S is the surface of E with positive orientation.

$z=4$

outward normal

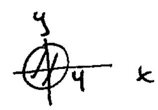


$\partial E = S$

S has two components and $\vec{F}$ is messy so use the Divergence Thm

{ S - piecewise smooth, outward normal, $\vec{F}$ defined on $\mathbb{R}^3$ - no holes }

$$\iint_S \vec{F}\cdot\vec{n}\,dS = \iint_S \vec{F}\cdot d\vec{S} \underset{\text{Div Thm}}{=} \iiint_E \operatorname{div}\vec{F}\,dV$$



$\operatorname{div}\vec{F} = \nabla\cdot\vec{F} = y^2 + x^2 + 2z$ so

$$\iiint_E \operatorname{div}\vec{F}\,dV = \iiint_E x^2+y^2+2z\,dV = \int_0^{2\pi}\int_0^4\int_r^4 (r^2+2z)\,r\,dz\,dr\,d\theta$$

cylindrical coords.

$$= \int_0^{2\pi}\int_0^4\int_r^4 (r^3+2rz)\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^4 \left[r^3z + rz^2\right]_r^4 dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^4 4r^3 + 16r - r^4 - r^3\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^4 3r^3 + 16r - r^4\,dr$$

$$= [\theta]_0^{2\pi}\left[\frac{3}{4}r^4 + 8r^2 - \frac{1}{5}r^5\right]_0^4$$

$$= 2\pi\cdot\left(\frac{3}{4}4^4 + \underbrace{8\cdot 4^2}_{2\cdot 4} - \frac{1}{5}4^5\right)$$

$$= 2\pi\left(5\cdot 4^3 - \frac{4^5}{5}\right) = \boxed{\frac{1152\pi}{5}}$$

/15

9. (15 points) Let $\mathbf{F}(x,y,z) = \langle x, x+y^3, x^2+y^2-z\rangle$ and let S be the surface $z = x^2 - y^2$ (saddle) for $x^2+y^2 \le 1$ with upward orientation and positively oriented boundary C. Find $\int_C \mathbf{F}\cdot d\mathbf{r}$.

could parameterize C but $\vec{F}$ is messy. $\vec{F}$ defined on $\mathbb{R}^3$ (no holes)

$$\int_C \vec{F}\cdot d\vec{r} \underset{\text{Stokes' Thm}}{=} \iint_S \operatorname{curl}\vec{F}\cdot d\vec{S}$$

(orientations ✓)

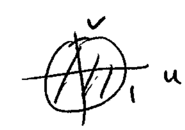
$$\operatorname{curl}\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & x+y^3 & x^2+y^2-z \end{vmatrix} = \langle 2y-0, -(2x-0), 1-0\rangle$$

$$= \langle 2y, -2x, 1\rangle$$

parameterize S: $\vec{r}(u,v) = \langle u, v, u^2-v^2\rangle$, $\underbrace{u^2+v^2 \le 1}_{D}$

$\vec{r}_u = \langle 1, 0, 2u\rangle$

$\vec{r}_v = \langle 0, 1, -2v\rangle$



$$\vec{r}_u\times\vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2u \\ 0 & 1 & -2v \end{vmatrix} = \langle -2u, 2v, 1\rangle$$

upward ✓

$$\iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = \iint_D \operatorname{curl}\vec{F}(\vec{r}(u,v))\cdot(\vec{r}_u\times\vec{r}_v)\,dA$$

$$= \iint_D \langle 2v, -2u, 1\rangle\cdot\langle -2u, 2v, 1\rangle\,dA = \iint_D -4uv - 4uv + 1\,dA$$

$u = r\cos\theta$, $v = r\sin\theta$, $|J| = r$

$0 \le \theta \le 2\pi$, $0 \le r \le 1$

$$= \int_0^{2\pi}\int_0^1 (-8r\cos\theta\cdot r\sin\theta + 1)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 (-8r^3\cos\theta\sin\theta + r)\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[-2r^4\cos\theta\sin\theta + \tfrac{1}{2}r^2\right]_0^1 d\theta = \int_0^{2\pi} -2\sin\theta\cos\theta + \tfrac{1}{2}\,d\theta$$

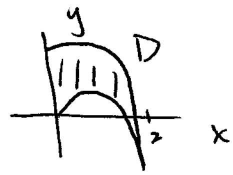
$$= \left[-\sin^2\theta + \tfrac{1}{2}\theta\right]_0^{2\pi} = \boxed{\pi}$$

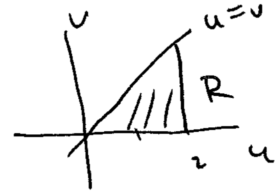
/15

10. (15 points) Find $\int_0^2 \int_{x-x^2}^{2-x^2} 6x\cos\left((x^2+y)^3\right)\,dy\,dx$ using the substitutions $u = x^2 + y$ and $v = x$.

$u = x^2 + y \quad \leadsto \quad x = v$

$v = x \qquad\qquad\quad y = u - v^2$





$0 \le x \le 2$

$0 \le v \le 2$

$x - x^2 \le y \le 2 - x^2$

$y = x - x^2 \qquad y = 2 - x^2$

$x^2 + y = x \qquad x^2 + y = 2$

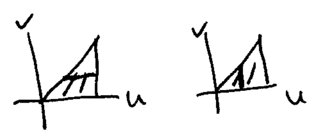
$v \le u \le 2$

$J = \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 0 & 1 \\ 1 & -2v \end{vmatrix} = -1$

so $|J| = 1$

$$\int_0^2 \int_{x-x^2}^{2-x^2} 6x\cos\left((x^2+y)^3\right) dy\,dx = \int_0^2 \int_v^2 6v\cos(u^3)\cdot 1\,du\,dv$$

$\uparrow$ hard so switch



$= \int_0^2 \int_0^u 6v\cos(u^3)\,dv\,du$

$= \int_0^2 \left[3v^2\cos(u^3)\right]_0^u du$

$= \int_0^2 3u^2\cos(u^3)\,du$

$= \left[\sin(u^3)\right]_0^2$

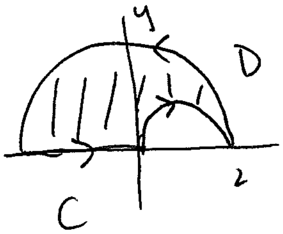
$= \sin(8) \cancel{- \sin(0)}$

$= \boxed{\sin(8)}$

/15

$\qquad\qquad\qquad\qquad\qquad\qquad F_1 \qquad\qquad F_2$

11. (15 points) Let $\mathbf{F}(x, y) = \langle xy + \cos(x^2), x - \arctan(y^2)\rangle$ and let D be the region of the plane above the x-axis inside the circle centered at the origin with radius 2 and outside the circle centered at $(1, 0)$ with radius 1. Let C be the boundary of D with positive orientation. Evaluate $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$.



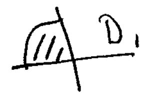
Again $\vec{F}$ is messy but nice on $\mathbb{R}^3$ – no holes

use Green's Thm.

By Green's Thm $\displaystyle\int_C \vec{F}\cdot d\vec{r} = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$

$\dfrac{\partial F_2}{\partial x} = 1 \qquad \dfrac{\partial F_1}{\partial y} = x \qquad$ So $\displaystyle\int_C \vec{F}\cdot d\vec{r} \overset{\text{Green's}}{=} \iint_D 1 - x \, dA$

use polar and split into two pieces



$D_1: \quad 0 \le r \le 2, \quad \frac{\pi}{2} \le \theta \le \pi$

$D_2: \quad (x-1)^2 + y^2 = 1 \qquad$ so $\quad 2\cos\theta \le r \le 2$

$x^2 - 2x + 1 + y^2 = 1 \qquad\qquad 0 \le \theta \le \frac{\pi}{2}$

$x^2 + y^2 = 2x$

$r^2 = 2r\cos\theta$

$r = 2\cos\theta$

$A(D_1) = \frac{1}{4}\pi(2)^2 = \pi$

$A(D_2) = \pi - \frac{1}{2}\pi(1)^2 = \frac{\pi}{2}$

$$\iint_D 1 - x\,dA = \iint_D 1\,dA - \iint_D x\,dA = A(D) - \iint_D x\,dA = \pi + \frac{\pi}{2} - \iint_D x\,dA$$

$$= \frac{3\pi}{2} - \left(\iint_{D_1} x\,dA + \iint_{D_2} x\,dA\right) = \frac{3\pi}{2} - \left(\int_{\frac{\pi}{2}}^{\pi}\int_0^2 r\cos\theta \cdot r\,dr\,d\theta + \int_0^{\frac{\pi}{2}}\int_{2\cos\theta}^{2} r\cos\theta \cdot r\,dr\,d\theta\right)$$

$$= \frac{3\pi}{2} - \int_{\frac{\pi}{2}}^{\pi}\cos\theta\,d\theta \int_0^2 r^2\,dr - \int_0^{\pi/2} \left[\frac{1}{3}r^3\cos\theta\right]_{2\cos\theta}^{2} d\theta$$

$\cos^2\theta \cdot \cos^2\theta$ double angle twice

$$= \frac{3\pi}{2} - \left(\sin\theta\right]_{\frac{\pi}{2}}^{\pi} \left(\frac{1}{3}r^3\right]_0^2 - \int_0^{\pi/2} \frac{8}{3}\cos\theta - \frac{8}{3}\cos^4\theta\,d\theta = \ldots = \frac{3\pi}{2} + \frac{8}{3} - \frac{8}{3} + \frac{\pi}{2} = \boxed{2\pi}$$

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